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PREFACE

The influence of force ratio, or numerical superiority, on victory in battle has been a subject for debate and conjecture for many years. One frequently hears that a 3-to-1 force ratio in the assault is necessary for, or (in other versions) sufficient to ensure, victory in land engagements. These statements are sufficiently common that an investigation of their validity is in order. ~~This~~ paper is a contribution to a determination of the extent to which victory in land battles is influenced by force ratio.( )

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1. The solid line on Figure 1 indicates the probability that an attacker will have the "advantage" in an engagement where his force ratio is as indicated on the abscissa. This curve was obtained by starting with the scatter-diagram of Figure 9 of Ref. 1. This scatter-diagram suggests that the defender's "advantage" (as defined in Ref. 1) in a battle with force ratio r is normally distributed with mean

$$\bar{V}_D = 0.115 - 0.367 \ln(r)$$

and standard deviation

$$\sigma = 0.297$$

where r is the ratio of initial attacker strength to intial defender strength. The numerical values in the preceeding formulae are from CORG-SP-128. The attacker's advantage, V_A, is by definition equal to the negative of the defender's advantage.

Using the foregoing, the probability that V_A will be positive can be calculated and plotted as a function of force ratio r.

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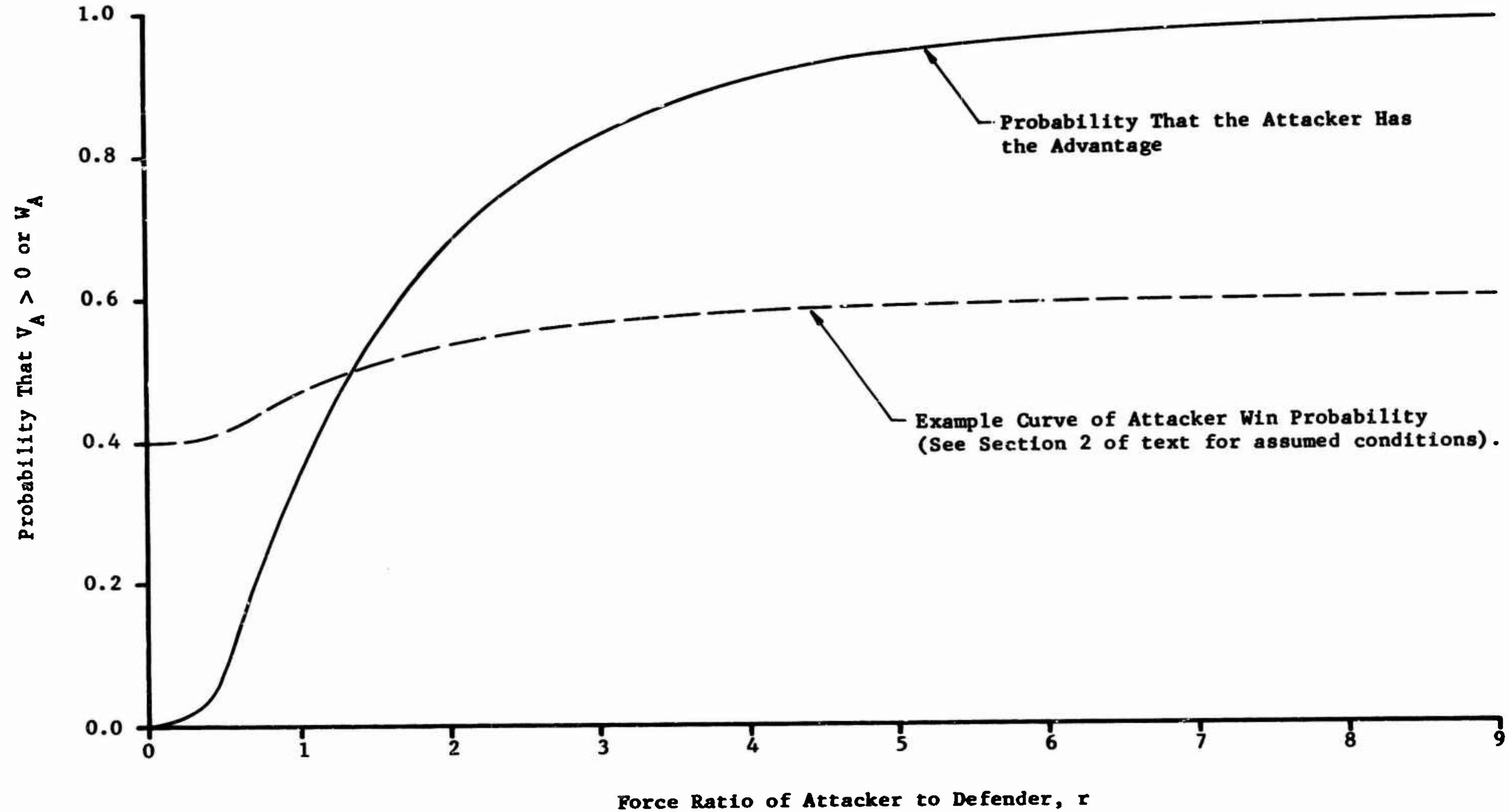

Figure 1: **PROBABILITY OF ATTACKER ADVANTAGE OR WIN <u>vs</u>. FORCE RATIO**

It is expressible as

$$f(r) = P(V_A > 0) = \Phi(\bar{V}_A/\sigma)$$

where

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \exp(-u^2/2)\,du$$

is the cumulative normal distribution function.

2. This does not tell the whole story, however, since there is no guarantee that $V_A > 0$ implies that the attacker is sure to win. An illuminating, though presumably oversimplified, model of this kind of uncertainty in the battle outcome can be obtained as follows. Let $W_A(W_D)$ stand for the event "attacker wins"("defender wins"). We suppose that the conditional probabilities of an attacker win, given $V_A > 0$ or $V_A < 0$, are constants (i.e., independent of the force ratio). Using obvious notation for these conditional probabilities, we can write

$$P(W_A) = P(W_A|V_A > 0)\ P(V_A > 0) + P(W_A|V_A < 0)\ P(V_A < 0)$$

$$= P(W_A|V_A < 0) + \left[P(W_A|V_A > 0) - P(W_A|V_A < 0)\right] \cdot P(V_A > 0)$$

where the last line follows from the preceeding one since $P(V_A < 0) = 1 - P(V_A > 0)$. The following assertions follow immediately from these equations.

a. $P(W_A) \equiv P(V_A > 0)$ for all values of r iff both

(1) $P(W_A|V_A > 0) = 1$, and

(2) $P(W_A|V_A < 0) = 0$.

b. $P(W_A) \equiv P(W_A|V_A < 0)$ for all values of r iff

$P(W_A|V_A < 0) = P(W_A|V_A > 0)$.

c. When $P(W_A|V_A > 0) - P(W_A|V_A < 0)$ is non-negative, then $P(W_A)$ is a monotonically increasing function of the force ratio, r, and rises from a minimum value of $P(W_A|V_A < 0)$ at $r \to 0$ to an asymptotic upper limit of $P(W_A|V_A > 0)$ as r tends to infinity. For instance, if $P(W_A|V_A < 0) = 0.4$ and $P(W_A|V_A > 0) = 0.6$, the resulting curve of $P(W_A)$ versus force ratio obtained by the equation above is as indicated by the dashed line on Figure 1.

3. As is apparent from the example shown, the impact of introducing these conditional probability values is to dilute the value of force ratio as a predictor of who will win. There are some data indicating that some such dilution must be present as shown in Table I prepared from data in Refs. 1 and 2. The table is compatible with the assumption that $P(W_A) = 0.5$ for all values of force ratio and case 2.b above applies, yielding

$$P(W_A|V_A < 0) = P(W_A|V_A > 0) = 0.5$$

Of course, other values of the conditional probabilities are also compatible with the tabulated data, and these lead to alternative (but not drastically different) interpretations.

Table I

Number of Battles Won By Winning Side and by Force Ratio Level

	FORCE RATIO LEVEL						TOTAL
	$0 \le r < 0.9$		$0.9 \le r < 1.5$		$1.5 \le r < \infty$		
W_A	28	(54%)	27	(48%)	39	(60%)	94
W_D	24	(46%)	29	(52%)	26	(40%)	79
TOTAL	52	(100%)	56	(100%)	65	(100%)	173

a. In particular, there is no evidence in Table I that a 3-to-1 force ratio is necessary for an attacker win to result.

b. Nor do the data support a contention that a 3-to-1 force ratio is sufficient for an attacker win. From data in SP-128 and SP-190 there are 15 battles with force ratios of at least 3-to-1 in favor of the attacker. Of these, the attacker won 10 (66%) and lost 5 (33%). However, for a binomial distribution with probability 0.5, a sample of size 15 would have a mean of 7.5 and a standard deviation of about 2, so that the observed value of 10 is only about 1.25 standard deviations out from the mean. Thus, these data are compatible with an assumed win probability of 0.5 even for force ratios of 3-to-1 and above.

4. Despite the repeated failure of these statistical tests to indicate a significant departure from a win probability of 0.5, maximum likelihood estimates of the win probability tend to be higher than

0.5 when the force ratio is favorable. This suggests that these "not significant" results may be due to the lack of discriminatory power associated with small-sized statistical samples. Willard (Ref. 3), dealing with initial strength data for 1493 of the battles recorded in Bodart's Kriegslexicon (Ref. 4), has also analyzed the dependence of victory on force ratio. However, Willard groups battles according to a "force ratio" which he defines as the ratio of the troop strength of the larger force to that of the smaller one. Accordingly, Willard's definition of "force ratio" (which we will always refer to as the "absolute force ratio" and denote by R to distinguish it from the attacker's force ratio, r) is related to ours through the equation

$$R = \mathrm{Max}\ (r,\ r^{-1}).$$

The numerically stronger force wins (symbolized by W_S) if $r > 1$ and W_A, or if $r < 1$ and W_D. To convert the solid curve of Figure 1 into a form suitable for use with Willard's data, we begin by letting f(r) stand for the function relating the attacker force ratio, r, to the values on the graph. If $R = r$, then the conditional probability that the numerically stronger force has the advantage (symbolized by $V_S > 0$) is:

$$P_+(V_S > 0) = f(R).$$

However, if $R = r^{-1}$, then the conditional probability that the numerically stronger force has the advantage is:

$$P_-(V_S > 0) = 1 - f(R^{-1}).$$

Table II

Willard Data

	Category I[a]			Category II[a]		
Range of R	No. Battles	Percent Won by the Stronger Side	95% Confidence Limits[b]	No. Battles	Percent Won by the Stronger Side	95% Confidence Limits[b]
1.0-1.5	473	58	53-63	64	61	47-73
1.5-2.0	251	65	59-71	51	72	57-84
2.0-2.5	122	75	67-83	51	82	68-92
2.5-3.0	58	57	43-71	38	87	73-97
3-4	56	73	59-84	69	75	65-85
4-5	30	63	44-80	37	81	64-91
5-6	9	89	56-99	38	95	84-99
6-7	13	77	48-93	22	95	76-100
> 7	9	67	29-87	102	91	85-97

[a]Category I battles are "open," described by Bodart as treffen, gefecht, and schlacht. Category II battles are "closed," described by Bodart as belagering, einnahme, ersturmung, capitulation, and uberfall.

[b]95 percent upper and lower confidence limits, based on each battle outcome representing an independent sample from a binomial distribution.

because for $R = r^{-1}$ the defender is the stronger side, and his advantage is the complement of the attacker's advantage. Hence, letting $P(R = r)$ stand for the probability that $R = r$, the unconditional probability that the numerically superior side has the advantage is:

$$P(V_S > 0) = f(R) \cdot P(R = r) + (1 - f(R^{-1})) \cdot P(R = r^{-1})$$

$$= f(R) \cdot P(R = r) + (1 - f(R^{-1})) \cdot (1 - P(R = r)).$$

The same argument applies mutatis mutandis to relate the probability that the numerically superior side wins to the probability $g(r)$ that the attacker wins an engagement in which he has the force ratio r, and yields the relation

$$P(W_S) = g(R) \cdot P(R = r) + (1 - g(R^{-1})) \cdot (1 - P(R = r)).$$

5. The conditional probabilities of winning, given the sign of attacker advantage provide ample free parameters for fitting Willard's data. If we put

$$a = P(W_A | V_A < 0), \quad \text{and}$$

$$a + b = P(W_A | V_A > 0),$$

then $P(W_A)$ can be expressed, using the relations in paragraph 2, as

$$g(r) = a + b\, f(r).$$

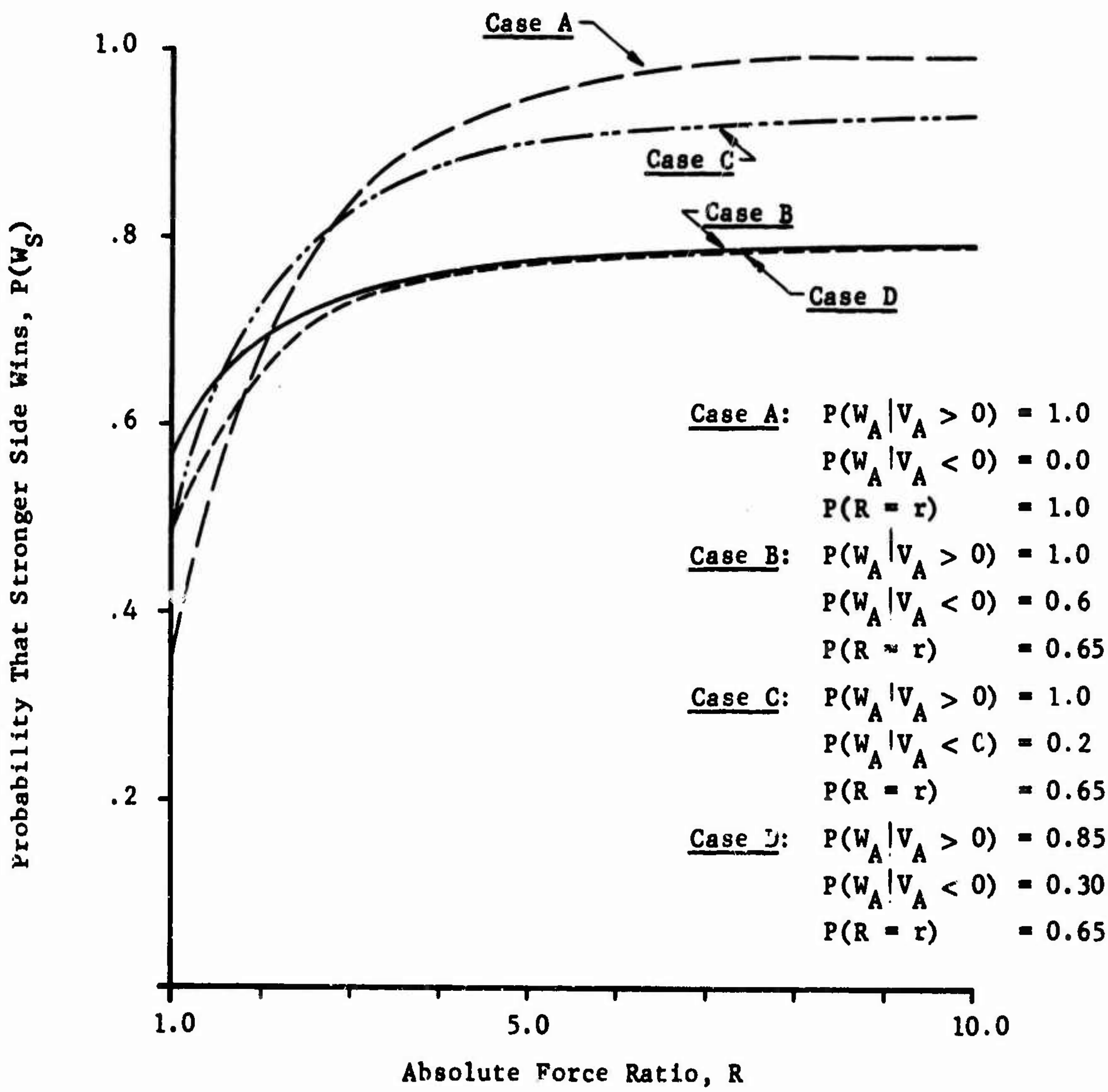

Figure 2: PROBABILITY OF WIN BY STRONGER SIDE FOR VARIOUS CASES

Substituting this expression for $g(\cdot)$ into the equation for $P(W_S)$ in paragraph 4 and simplifying yields

$$P(W_S) = 1 - P(R = r) + a(2P(R = r) - 1)$$

$$+ b\left[P(V_S > 0) + P(R = r) - 1\right]$$

where $P(V_S > 0)$ is equivalent to the curve for case A on Figure 2.

For the battles analyzed in Refs. 1 and 2, the observed fraction of battles for which $R = r$ (i.e., for which the attacker had the larger force) is 0.65. From the same sources, we observe that the attacker won 66 out of 78 battles in which he had the advantage, or about 84 percent. Also, the attacker won 29 out of 95 battles in which the defender had the advantage, or about 31 percent. In each instance, the values quoted are based on a mix of Category I and Category II battles, but with a sizable majority of Category I battles. Figure 3 shows a comparison of Willard's data with the model just developing, using SP-128 data for the basic prediction curve. The degree of agreement between SP-128 predictions and Willard's data seems to me to be acceptable.

No such base for predicting Willard's Category II data is afforded by Refs. 1 and 2. Figure 4 shows a curve fitted to Willard's data by trial-and-error, starting from an initial assumption that $P(R = r) = 0.65$, the value typical of the Refs. 1 and 2 battles. This led to a choice of 1.00 for $P(W_A | V_A > 0)$, and 0.20 for $P(W_A | V_A < 0)$. A practically identical curve results from chosing $P(R = r) = 0.90$, $P(W_A | V_A > 0) = 0.95$, $P(W_A | V_A < 0) = 0.25$. This illustrates both that there is, in the model, an ample supply of parameters that may be adjusted, and

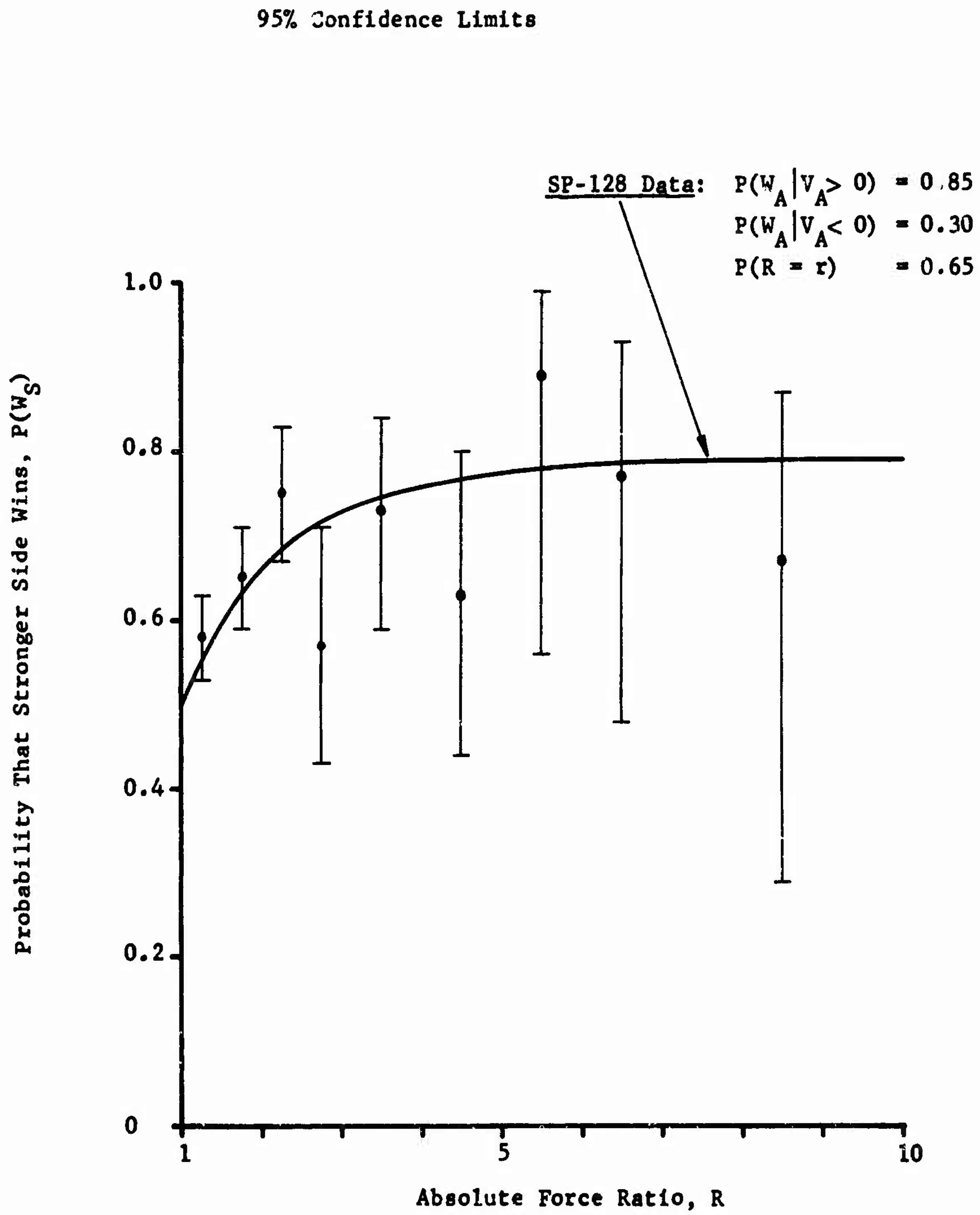

Figure 3: PROBABILITY OF WIN BY STRONGER SIDE IN CATEGORY I (OPEN) BATTLES

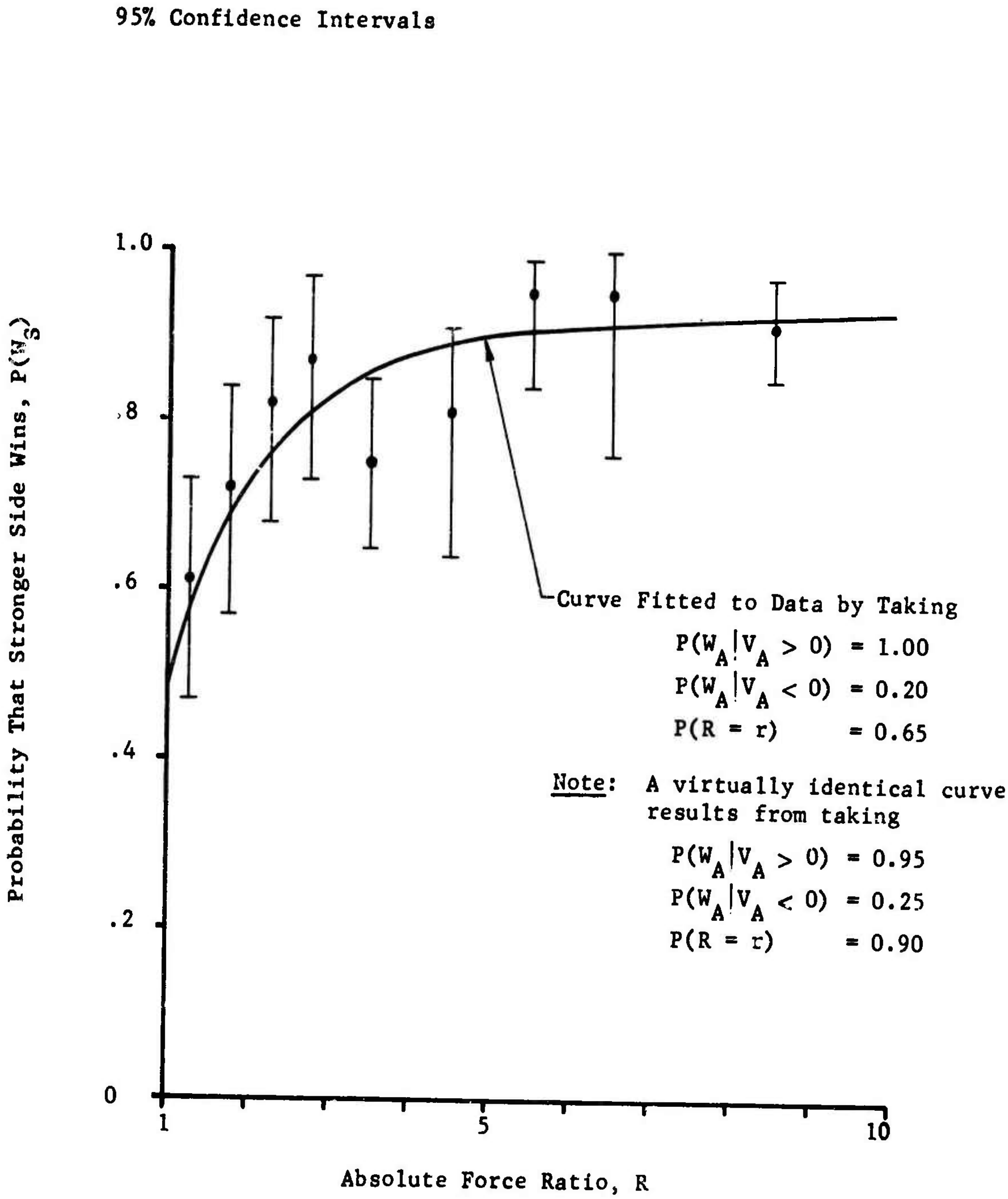

Figure 4: PROBABILITY OF WIN BY STRONGER SIDE IN CATEGORY II (CLOSED) BATTLES

that certain data sets may be unable to distinguish among different choices of parameters. It would seem to be possible to rather easily obtain an estimate of P(R = r) by determining this datum from other sources for either a partial or an exhaustive sample of Willard's Category II battles, and estimating P(R = r) from the sample.

6. These results seem to suggest the following ideas:

a. There are (at least) two categories of battles that can be distinguished through the analysis of historical combat data. These may be called "open," signifying that each side can, with roughly equal facility, break contact and withdraw; and "closed," indicating that one side would have a markedly easier time of withdrawing on its own initiative than would its opponent.

b. These two categories can be represented numerically by assigning suitable values to the conditional probability that the stronger side will be able to capitalize on its "advantage."

c. The predictive procedure implied by the preceeding constitutes a considerably improved refinement of the "3-to-1" numerical superiority" doctrine.

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